

A fundamental length as a candidate for dark energy: a DSR inspired FRW spacetime

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Abstract

We show that the existence of a fundamental length, introduced in Deformed Special Relativity (DSR) inspired minisuper (phase-) space, causes the behavior of the scale factor of the universe to change from that of a universe filled with dust to an accelerating universe driven by a cosmological constant.

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1 Introduction

The question of the evolution of cosmos from the beginning to its present stage constitutes one of the most important questions in the history of science. More precisely, the question may be asked as to how the present accelerating phase of the universe could result from the big-bang. To answer this question, a heap of observational data have been collected and a huge amount of theoretical work has been done. To describe the behavior of the present accelerating phase of the universe discovered a few years ago, numerous models have been devised and introduced [1]. A common tool used to describe the accelerated expansion of our universe is what is known as dark energy which is the basic ingredient in model theories with a cosmological constant.

It is generally believed that quantization of gravity could lead to the removal of singularities which one encounters in classical general relativity. On the other hand it is also generally believed that the existence of a fundamental length is a natural feature in all theories that endeavor to answer the old and interesting question of how to quantize gravity [2, 3]. This fundamental length has been introduced in some model theories by hand [4, 5, 6, 7, 8, 9, 10, 11, 12] where it is shown that such effective theories can be recovered as the limit of full quantum gravity [4, 5, 13, 14, 15]. Therefore, the motivation behind the construction of these effective models is to study the effects of such a fundamental length scale on simple scenarios which are exactly solvable. One of these approaches is to modify or deform the algebraic structure of the phase-space. Such deformations may be done in various manners, a few example of which can be found in [16, 17, 18, 19, 20, 21, 22].

To study the effects of the existence of a fundamental length in a cosmological scenario, one can construct a model based on the noncommutative structure of the Deformed (Doubly) Special Relativity [5] which is related to what is known as the κ -deformation [23, 24]. This way of introducing noncommutativity is interesting because of its compatibility with Lorentz symmetry [23, 24, 25]. The κ -deformation is introduced and studied in [26, 27, 28]. The κ -Minkowski space [28, 29] arises

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naturally from the κ -Poincare algebra [26, 27] such that the ordinary brackets between the coordinates are replaced by

$$\{x_0, x_i\} = \frac{1}{\kappa} x_i, \quad (1)$$

where $\{, \}$ represents the Poisson bracket and κ is the deformation (noncommutative) parameter which has the dimension of mass $\kappa = \epsilon \ell^{-1}$ when $c = \hbar = 1$, where $\epsilon = \pm 1$ [30] such that κ and ℓ are interpreted as dimensional parameters that identify with the fundamental energy and length, respectively. In the following we restrict ourselves to the sector $\epsilon = -1$. This fundamental length can be identified, for example, with the Planck length. As mentioned above, one can change the structure of the phase-space based on equation (1). To study the effects of this kind of deformation, we briefly review the Hamiltonian formalism of the FRW spacetimes in the next section. In section 3, the κ -deformed phase-space for FRW spacetimes is introduced and its effects are studied. The paper ends with a discussion of the results.

2 Phase-space structure of the FRW spacetimes

Let us start by briefly studying the ordinary FRW model

$$ds^2 = -N^2(t)dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (2)$$

where $N(t)$ is the lapse function. The Einstein-Hilbert Lagrangian with a general energy density $V(a)$ becomes

$$\begin{aligned} \mathcal{L} &= \sqrt{-g}(R[g] - V(a)) \\ &= -6N^{-1}a\dot{a}^2 - Na^3V(a), \end{aligned} \quad (3)$$

where $R[g]$ is the Ricci scalar and in the second line the total derivative term has been ignored. The corresponding Hamiltonian up to a sign becomes

$$\mathcal{H}_0 = \frac{1}{24}Na^{-1}p_a^2 - Na^3V(a). \quad (4)$$

Here, we note that since the momentum conjugate to $N(t)$, $\pi = \frac{\partial \mathcal{L}}{\partial N}$ vanishes, the term $\lambda\pi$ must be added as a constraint to Hamiltonian (4). The Dirac Hamiltonian then becomes

$$\mathcal{H} = \frac{1}{24}Na^{-1}p_a^2 - Na^3V(a) + \lambda\pi. \quad (5)$$

Now, the equations of motion with respect to the above Hamiltonian are given by

$$\begin{aligned} \dot{a} &= \{a, \mathcal{H}\} = \frac{1}{12}Na^{-1}p_a, \\ \dot{p}_a &= \{p_a, \mathcal{H}\} = \frac{1}{24}Na^{-2}p_a^2 + 3Na^2V(a) + Na^3V'(a), \\ \dot{N} &= \{N, \mathcal{H}\} = \lambda, \\ \dot{\pi} &= \{\pi, \mathcal{H}\} = -\frac{1}{24}a^{-1}p_a^2 + a^3V(a), \end{aligned} \quad (6)$$

where a prime represents differentiation with respect to the argument. Note that to satisfy the constraint $\pi = 0$ at all times the secondary constraint $\dot{\pi} = 0$ must also be satisfied. A simple calculation leads to

$$\dot{a} = \sqrt{\frac{1}{6}a^2V(a)}, \quad (7)$$

where in the above we fix the gauge by taking $N = 1$, that is, we work in the comoving gauge. Note that the other equations will automatically become consistent if the above equation is satisfied. The solution for dust for which $V(a) = \rho_0 a^{-3}$ becomes

$$a(t) = \frac{3^{\frac{1}{3}}}{2} (\sqrt{\rho_0} t + C_1)^{\frac{2}{3}}, \quad (8)$$

where C_1 is the integration constant which is set to zero. This scale factor shows a singularity at $t = 0$, the so called big-bang singularity, and has a power-law behavior which is not consistent with late time observations.

3 κ -deformed phase-space structure of the FRW spacetimes

It has long been argued that the deformation in phase-space can be seen as an alternative path to quantization, based on Wigner quasi-distribution function and Weyl correspondence between quantum-mechanical operators in Hilbert space and ordinary c-number functions in phase space, see for example [31] and the references therein. The deformation in the usual phase-space structure is introduced by Moyal brackets which are based on the Moyal product [16, 17, 18, 19]. However, to introduce such deformations it is more convenient to work with Poisson brackets rather than Moyal brackets.

From a cosmological point of view, models are built in a minisuper-(phase)-space. It is therefore safe to say that studying such a space in the presence of the deformations mentioned above can be interpreted as studying the quantum effects on cosmological solutions. One should note that in gravity (here, cosmology) the effects of quantization are woven into the existence of a fundamental length [2], as mentioned in the introduction. The question then arises as to what form of deformations in phase-space is appropriate for studying quantum effects in a cosmological model? Studies in noncommutative geometry [10, 11, 12] and generalized uncertainty principle (GUP) [32] have been a source of inspiration to answer the above question. More precisely, introduction of modifications in the structure of geometry in the way of noncommutativity [10, 11, 12] has become the basis from which similar modifications in phase-space have been inspired. In this approach, the fields and their conjugate momenta play the role of coordinate basis in noncommutative geometry [33, 34]. In doing so an effective model is constructed whose validity will depend on its power of prediction. For example, if in a model field theory the fields are taken as noncommutative, as has been done in [33, 34], the resulting effective theory predicts the same Lorentz violation as a field theory in which the coordinates are considered as noncommutative [35, 36, 37]. Over the years, a large number of works on noncommutative fields [16, 17, 18, 19] have been inspired by noncommutative geometry model theories [10, 11, 12]. As a further example, it is well known that string theory can be used to suggest a modification in the bracket structure of coordinates, also known as GUP [32] which is used to modify the phase-space structure [20].

In this paper we will examine a new kind of modification in the phase-space structure inspired by relation (1), much the same as what has been done in [16, 20, 22]. In what follows we introduce non-commutativity based on κ -Minkowskian space and study its consequences on the solutions discussed in the previous section. To introduce noncommutativity one can start with

$$\{\hat{N}(t), \hat{a}(t)\} = -\ell \hat{a}(t). \quad (9)$$

This is similar to equation (1) since one can interpret $N(t)$ and $a(t)$, appearing as the coefficients of dt and $d\vec{x}$, in the same manner as x_0 and x_i appearing in (1) respectively. For this reason we shall call it the κ -Minkowskian-minisuper-phase-space. In this case the Hamiltonian becomes

$$\hat{\mathcal{H}}_0 = \frac{1}{24} \hat{N} \hat{a}^{-1} \hat{p}_a^2 - \hat{N} \hat{a}^3 V(\hat{a}), \quad (10)$$

where the ordinary Poisson brackets are satisfied except in (9). To progress further, one introduces the following variables [38]

$$\begin{cases} \hat{N}(t) = N(t) + \ell a(t)p_a(t), \\ \hat{a}(t) = a(t), \\ \hat{p}_a(t) = p_a(t). \end{cases} \quad (11)$$

It can be easily checked that the above variables satisfy (9) if the unprimed variables satisfy ordinary Poisson brackets. With the above transformations, Hamiltonian (10) changes to

$$\mathcal{H}_0^{nc} = \frac{1}{24}Na^{-1}p_a^2 - Na^3V(a) + \frac{1}{24}\ell p_a^3 - \ell a^4V(a)p_a. \quad (12)$$

Obviously, the momentum conjugate to $N(t)$ does not appear in the above Hamiltonian, that is, π is a primary constraint. It can be checked by using Legendre transformations that the conjugate momentum corresponding to $N(t)$, $\pi = \frac{\partial \mathcal{L}}{\partial \dot{N}}$, vanishes. It is therefore necessary to add the term $\lambda\pi$ to Hamiltonian (12) to obtain the Dirac Hamiltonian

$$\mathcal{H}^{nc} = \frac{1}{24}Na^{-1}p_a^2 - Na^3V(a) + \frac{1}{24}\ell p_a^3 - \ell a^4V(a)p_a + \lambda\pi. \quad (13)$$

The equations of motion with respect to Hamiltonian (13) are

$$\begin{aligned} \dot{a} &= \{a, \mathcal{H}^{nc}\} = \frac{1}{12}Na^{-1}p_a + \frac{1}{8}\ell p_a^2 - \ell a^4V(a), \\ \dot{p}_a &= \{p_a, \mathcal{H}^{nc}\} = \frac{1}{24}Na^{-2}p_a^2 + 3Na^2V(a) + Na^3V'(a) + 4\ell a^3V(a)p_a + \ell a^4V'(a)p_a, \\ \dot{N} &= \{N, \mathcal{H}^{nc}\} = \lambda, \\ \dot{\pi} &= \{\pi, \mathcal{H}^{nc}\} = -\frac{1}{24}a^{-1}p_a^2 + a^3V(a), \end{aligned} \quad (14)$$

where a prime denotes differentiation with respect to the argument. Again, we restrict ourselves to the comoving gauge for which $N = 1$. We also note that the secondary constraint, $\dot{\pi} = 0$, must be satisfied in order that the primary constraint, namely $\pi = 0$, is satisfied at all times. If p_a is now calculated from the secondary constraint, $\dot{\pi} = 0$, and the result is substituted in the first equation in (14) one obtains

$$\dot{a} - 2\ell a^4V(a) = \sqrt{\frac{1}{6}a^2V(a)}. \quad (15)$$

It is easy to check that the above equation is consistent with the second equation in (14) as well. Note that this equation reduces to the commutative case (7) when $\ell \rightarrow 0$. The exact solution for $V(a) = \rho_0 a^{-3}$, representing dust, becomes

$$a(t) = \frac{1}{2 \times 3^{\frac{1}{3}}} \left[\frac{-1 + C_2 e^{3\ell \rho_0 t}}{\ell \sqrt{\rho_0}} \right]^{\frac{2}{3}}, \quad (16)$$

where C_2 is an integration constant which must satisfy $C_2 e^{3\ell \rho_0 t} \geq 1$. In the following we set $C_2 = 1$. We note that the limit $\ell \rightarrow 0$ still coincides with the commutative result (8), predicting a big-bang singularity. The scale factor (16) with the chosen integration constant in the limit $\ell \rightarrow 0$ then leads to

$$a(t) = \frac{3^{\frac{1}{3}}}{2} (\sqrt{\rho_0} \ t + \mathcal{O}(\ell))^{\frac{2}{3}}. \quad (17)$$

This means that the scale factor begins from a big-bang singularity and then behaves as if the universe is filled with dust. However, the difference between the commutative and noncommutative cases are

manifest in the late time behavior where in the noncommutative case with $t \rightarrow \infty$, the scale factor becomes

$$a(t) \propto e^{2\ell\rho_0 t}, \quad (18)$$

for which (2) represents a de-Sitter metric with a cosmological constant $\Lambda = 12\ell^2\rho_0^2$. Therefore, for late times the behavior of the scale factor becomes exponential, pointing to the existence of a fundamental length which can be interpreted as a candidate for dark energy. Taking the relation for Λ given above, the value of the cosmological constant can be estimated. If one chooses $\ell = \ell_P$ where $\ell_P \sim 10^{-35}m$ is the Planck length and the matter density is taken as $\rho_0 = 1.9 \times 10^{-29}h^2 gcm^{-3}$ [39] where h is the Planck constant, the cosmological constant is found to be $\Lambda \sim 10^{-170}m^{-2}$ which is well within the suggested upper bound $\Lambda_{upperbound} \sim 10^{-52}m^{-2}$, a generally accepted value determined by observation [40]. Note that the value of the cosmological constant in our model is much smaller than the suggested upper bound. This value could be improved to become more in line with the observational upper bound if other factors such as radiation are taken into account. This means that the scale factor begins with a dust-like behavior before entering the accelerating phase (cosmological constant domination) which is a direct consequence of the existence of a fundamental length.

4 Discussion

The construction of an effective theory is usually the result of the observation that the the corresponding full theory is too complicated to deal with. This, for example, is true in describing the quantum effects on cosmology since the full theory is immensely difficult to handle [41]. The DSR can be interpreted as one such effective theory. In the present study, we have introduced a fundamental length by employing an effective theory, namely a DSR-inspired model. In doing so we have chosen the simplest cosmological model, namely a FRW spacetime in the presence of matter in the form of dust. Here, the introduction of a fundamental length causes additional terms to appear in Hamiltonian (13) as compared to (5). As has been mentioned in [22, 31, 42], these extra terms can be interpreted as the effects of high energy corrections of a full theory, e.g., the string theory¹. On the other hand, our results can be used to address one of the problems of interest in cosmology, i.e. dark energy. It is therefore reasonable to assume that the modification introduced through relation (9) may also be relevant in model theories dealing with scenarios involving dark energy and the accelerating universe.

In this paper, we have shown that the existence of a fundamental length can play the role of a cosmological constant at late times. This means that such a notion can be employed to account for dark energy at late times. Note that the cosmological constant is only a parameter which is introduced by hand to describe the accelerating behavior of the universe. Here however, its appearance at late times is the direct consequence of the existence of the fundamental length. Although we have introduced such a length by hand here, its existence has its roots in quantum gravity [2], considered as the full theory. It therefore seems plausible that the existence of a fundamental length can describe the present accelerating behavior of the universe.

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¹This interpretation is consistent with the one introduced at the beginning of section 3 in that the extra terms can be interpreted as quantum effects.

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